Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021–8928(00)00111-8

J. Appl. Maths Mechs, Vol. 64, No. 5, pp. 833–843, 2000

0021–8928/00/$—see front matter

APPLICATIONS OF A FAMILY OF LYAPUNOV FUNCTIONS†

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(*Received* 17 *November* 1999)

A procedure for constructing a family of Lyapunov functions, which enable one to obtain criteria for asymptotic stability in the first approximation, is proposed for the systems of differential equations for the motion of a material system of very general form. The problem of constructing control systems which ensure asymptotic stability "in the large" and the problem of improving the quality of the transient are considered. Examples are given of the use of the proposed procedure in the problem of stabilizing the planar motions of a satellite in an elliptic orbit and in the problem of stabilizing the programmed motion of a mathematical pendulum with a moving suspension point and of a gyropendulum on a moving base. © 2001 Elsevier Science Ltd. All rights reserved.

A family of Lyapunov functions for investigating the stability "in the small" of the perturbed motion of mechanical systems has been constructed in [1]. This approach was used in [2] to construct mechanical systems which possess a programmed motion which is asymptotically stable "on the whole".

Below, these results are extended to a wider class of material systems which, apart from mechanical systems, also includes systems of a non-mechanical nature. Moreover, the family of Lyapunov functions which is constructed here is used to improve the quality of the transient in mechanical systems by means of optimal control.

1. CONSTRUCTION OF A FAMILY OF LYAPUNOV FUNCTIONS FOR INVESTIGATING STABILITY IN THE FIRST APPROXIMATION

Suppose the perturbed motion of a material system is described by the following vector equations

$$A_1(x_1,\dot{x}_1,x_2,t)\ddot{x}_1 = B_1(x_1,\dot{x}_1,x_2,t), \quad N_1(x_1,\dot{x}_1,x_2,t)\dot{x}_2 = K_1(x_1,\dot{x}_1,x_2,t) \tag{1.1}$$

where A_1 and N_1 are $n \times n$ and $m \times m$ matrices, x_1 and B_1 are n-dimensional vectors and x_2 and K_1 are m-dimensional vectors.

The elements A_1 and N_1 and of the vectors B_1 and K_1 are assumed to be bounded, continuous and continuously differentiable in a certain bounded domain $G\{x, x\}$ which includes $x_1 = 0, x_1 = 0, x_2 = 0$ when $t \geqslant t_0$. It is additionally assumed that $\det^2|A_1| > \delta_1$, $\det^2|N_1| > \delta_2$ in the domain G when $t \geqslant t_0$, where δ_1 and δ_2 are certain positive constants. These conditions are necessary in to order to ensure the existence and uniqueness of the solutions of Eqs (1.1) in the domain G when $t \geqslant t_0$ and for other objectives which are attained below.

The problem involves the construction of Lyapunov functions such that these functions themselves and their time derivatives form a certain family of quadratic forms in x_1, x_2 with matrices with diagonal elements which are not identically zero and contain not only the A_1, B_1, N_1, K_1 occurring in (1.1) but also derivatives of the vector functions $f(x_1, t)$.

The meaning of the formulation of this problem lies in the fact that its solution opens up the possibility of obtaining an entire family of criteria for the asymptotic stability of the trivial solution of system (1.1) solely using the generalized Silvester's criteria [3] in which, in addition to the parameters of the system, the arbitrary functions $f(x_1, t)$ also occur.

In the following sections of this paper, these Lyapunov functions will be used to construct control systems which ensure asymptotic stability "in the large" and "on the whole" as well as to evaluate and improve the quality of the transient.

Several of these problems have been solved previously [1, 2] in the case of mechanical systems with a choice of the function $f(x_1, t)$ in the linear form $H(t)\, x_1$.

†*Prikl. Mat. Mekh.* Vol. 64, No. 5, pp. 869–880, 2000.

Note that, unlike mechanical systems, the matrices A_1 and N_1 here depend on x_1 and cannot be symmetric and positive-definite.

In order to symmetrize these matrices and make them positive-definite, equations (1.1) can be multiplied by the transposed matrices A_1^T and N_1^T respectively. When this is done, Eqs (1.1) take the form

$$A(x_1,\dot{x}_1,x_2,t)\ddot{x}_1 = B(x_1,\dot{x}_1,x_2,t), \quad N(x_1,\dot{x}_1,x_2,t)\dot{x}_2 = K(x_1,\dot{x}_1,x_2,t) \tag{1.2}$$

where $A = A_1^T A_1$, and $N = N_1^T N_1$ are positive-definite and symmetric matrices and $B = A_1^T B_1$ and $K = N_1^T R_1$ are n- and m-dimensional vectors, respectively.

We now make the substitution

$$\dot{x}_1 = y + f(x_1, t), \quad f(0, t) = 0 \tag{1.3}$$

where $f(x_1, t)$ is an arbitrary n-dimensional vector function with bounded and differentiable elements which admits of an infinitesimal higher limit. On multiplying the first equation of (1.2) scalarly by the vector $y = x_1 - f(x_1, t)$ and the second equation by the vector x_2 and then adding them, we obtain equations, which after making the substitution

$$\ddot{x}_1 = \dot{y} + \frac{\partial f}{\partial x_1} y + \frac{\partial f}{\partial x_1} f(x_1,t) + \frac{\partial f}{\partial t}$$

and using the equality

$$y^T A\dot{y} = \frac{1}{2}\frac{d}{dt}(y^T Ay) - \frac{1}{2} y^T \frac{dA}{dt} y$$

reduce to the form

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(y^T A\dot{y} + x_2^T Nx_2) &= y^T B - y^T A\frac{\partial f}{\partial x_1} y - y^T A\left(\frac{\partial f}{\partial x_1} f + \frac{\partial f}{\partial t}\right) + \\ &+ \frac{1}{2} y^T \frac{dA}{dt} y + x_2^T K + \frac{1}{2} x_2^T \frac{dN}{dt} x_2 \end{aligned} \tag{1.4}$$

We will assume the B, K, and f are expandable in convergent power series in x_1, x_1, x_2. The terms of the first approximation of these vectors are expressed as follows:

$$f = \left(\frac{\partial f}{\partial x_1}\right)_0 x_1, \quad B = \left(\frac{\partial B}{\partial \dot{x}_1}\right)_0 \dot{x}_1 + \left(\frac{\partial B}{\partial x_1}\right)_0 x_1 + \left(\frac{\partial B}{\partial x_2}\right)_0 x_2$$

$$K = \left(\frac{\partial K}{\partial \dot{x}_1}\right)_0 \dot{x}_1 + \left(\frac{\partial K}{\partial x_1}\right)_0 x_1 + \left(\frac{\partial K}{\partial x_2}\right)_0 x_2$$

Then

$$y^T B = y^T\left(\frac{\partial B}{\partial \dot{x}_1}\right)_0 \dot{x}_1 + y^T\left(\frac{\partial B}{\partial x_1}\right)_0 x_1 + x_2^T\left(\frac{\partial B}{\partial x_2}\right)_0^T y$$

$$x_2^T K = \dot{x}_1^T\left(\frac{\partial K}{\partial \dot{x}_1}\right)_0^T x_2 + x_1^T\left(\frac{\partial K}{\partial x_1}\right)_0^T x_2 + x_2^T\left(\frac{\partial K}{\partial x_2}\right)_0 x_2$$

Retaining terms of no higher than second order infinitesimals on the right-hand side of Eqs (1.4) and replacing the vector y on the right-hand side by $[\dot{x}_1 - H(t)x_1]$, where $H(t) = (\partial f/\partial x_1)_0$, we obtain

$$\frac{1}{2}\frac{d}{dt}(y^T Ay + x_2^T Nx_2) = -\dot{x}_1^T\Gamma\dot{x}_1 - \dot{x}_1^T(\Gamma_1 + \Gamma H + \Gamma^T H)x_1 - x_1^T H^T(\Gamma H - \Gamma_1)x_1 +$$

$$+x_2^T\left[\left(\frac{\partial K}{\partial x_2}\right)_0+\frac{1}{2}\left(\frac{dN}{dt}\right)_0\right]x_2+\dot{x}_1^T\left(\frac{\partial K}{\partial \dot{x}_1}\right)_0^T x_2-x_1^T H^T\left(\frac{\partial K}{\partial \dot{x}_1}\right)_0^T x_2+$$
$$+x_2^T\left(\frac{\partial B}{\partial x_2}\right)_0^T \dot{x}_1-x_2^T\left(\frac{\partial B}{\partial x_2}\right)_0^T Hx_1 \tag{1.5}$$

where

$$\Gamma=AH-\left(\frac{\partial B}{\partial \dot{x}_1}\right)_0-\frac{1}{2}\left(\frac{dA}{dt}\right)_0$$
$$\Gamma_1=A(H^2+\dot{H})-\left(\frac{\partial B}{\partial \dot{x}_1}\right)_0 H-\left(\frac{\partial B}{\partial x_1}\right)_0$$

From the matrices of the first and second terms on the right-hand side of Eq. (1.5), we separate out the symmetric and skew symmetric parts

$$\Gamma=\mathrm{D}+G, \Gamma_1+\Gamma H+\Gamma^T H=C+E$$

where D and C are the symmetric and G and E are the skew symmetric parts of the matrices. Transferring the term $x_1^T Cx_1$ to the left-hand side, we obtain

$$\frac{1}{2}\frac{d}{dt}V=-\dot{x}_1^T D\dot{x}_1-\dot{x}_1^T Ex_1-x_1^T\left[H^T(\Gamma H-\Gamma_1)-\frac{\dot{C}}{2}\right]x_1+$$
$$+x_2^T\left[\left(\frac{\partial K}{\partial x_2}\right)_0+\frac{1}{2}\left(\frac{dN}{dt}\right)_0\right]x_2+\dot{x}_1^T\left[\left(\frac{\partial K}{\partial \dot{x}_1}\right)_0^T+\left(\frac{\partial B}{\partial x_2}\right)_0\right]x_2-\dot{x}_1^T H^T\left[\left(\frac{\partial K}{\partial \dot{x}_1}\right)_0^T+\left(\frac{\partial B}{\partial x_2}\right)_0\right]x_2 \tag{1.6}$$
$$V=\frac{1}{2}(y^T Ay+x_2^T Nx_2+x_1^T Cx_1)$$

Here, the following relations have been taken into account

$$\dot{x}_1^T G\dot{x}_1\equiv 0,\quad \dot{x}_1^T Cx_1=\frac{1}{2}\frac{d}{dt}(\dot{x}_1^T Cx_1)-\frac{1}{2}x_1^T\dot{C}x_1$$

The matrix H has n^2 arbitrary elements. Consequently, the function V, which depends on them, can be considered as the required class of Lyapunov functions since the right-hand side of equality (1.6), which is the total derivative of the function V with respect to t, is a quadratic form with respect to x_1, x_1, x_2 with a matrix, the diagonal elements of which are not identically equal to zero. This is impossible to achieve when $H\equiv 0$.

There is therefore the possibility of obtaining the entire family of criteria for the asymptotic stability of the unperturbed motion in the first approximation by solely making use of the generalized Silvester's criteria [3].

2. THE CONSTRUCTION OF SYSTEMS WITH A PROGRAMMED MOTION WHICH IS ASYMPTOTICALLY STABLE "IN THE LARGE"

We introduce the control vectors Q_1 and Q_2 into the right-hand side of system (1.2) and obtain

$$\begin{aligned}A(x_1,\dot{x}_1,x_2,t)\ddot{x}_1&=B(x_1,\dot{x}_1,x_2,t)+Q_1\\ N(x_1,\dot{x}_1,x_2,t)\dot{x}_2&=K(x_1,\dot{x}_1,x_2,t)+Q_2\end{aligned} \tag{2.1}$$

When x_1 is replaced by y using formula (1.3), scalar multiplication of the first equation of (2.1) by y and the second equation by x_2 leads to the following analogue of Eq. (1.4)

$$\frac{1}{2}\frac{d}{dt}(y^TAy+x_2^TNx_2)= \\ =y^T\left\{Q_1+B-A\left(\frac{\partial f}{\partial x_1}\right)y-A\left[\left(\frac{\partial f}{\partial x_1}\right)f+\frac{\partial f}{\partial t}\right]+\frac{1}{2}\frac{dA}{dt}y\right\}+x_2^T\left(K+\frac{1}{2}\frac{dN}{dt}x_2+Q_2\right) \tag{2.2}$$

If the vectors Q_1 and Q_2 are chosen in the form

$$Q_1=-Dy-F_1x_1-B+A\left(\frac{\partial f}{\partial x_1}\right)y+A\left[\left(\frac{\partial f}{\partial x_1}\right)f+\frac{\partial f}{\partial t}\right]-\frac{1}{2}\frac{dA}{dt}y \\ Q_2=-K-\frac{1}{2}\frac{dN}{dt}x_2-F_2x_2 \tag{2.3}$$

we then obtain

$$\frac{1}{2}\frac{d}{dt}(y^TAy+x_2^TNx_2+x_1^TF_1x_1)=-y^TDy+\left(f^TF_1+x_1^T\frac{\dot{F}_1}{2}\right)x_1-x_2^TF_2x_2 \tag{2.4}$$

where D, F_1, F_2 are symmetric, positive-definite matrices.

The function $y^T Ay + x_2^T Nx_2 + x_1^T F_1x_1$, constructed as a positive-definite matrix for all x_1, x_1, x_2, t, admits of an infinitesimal higher limit and an infinitely large lower limit. Consequently, when the function $(f^T F_1 + x_1^T F_1/2)x_1$ is negative-definite, the right-hand side of Eq. (2.4) will be negative-definite with respect to $x_1, \dot{x}_1, x_2$ and, in this case, the programmed state $x_1 = \dot{x}_1 = 0, x_2 = 0$ of system (2.1) will be stabilized "in the large". If this approach is applied to the stabilization of a preset state "on the whole" then certain additional conditions have to be borne in mind.

We will now explain these conditions. For this purpose, we consider the right-hand sides of Eqs. (2.3) which have the terms

$$\frac{1}{2}\frac{dA}{dt}y \text{ and } \frac{1}{2}\frac{dN}{dt}x_2$$

containing the highest derivatives x_1 and x_2.

Note that these derivatives do not occur in the equations of the first approximation since the terms containing them are of a high order of smallness. Consequently, when there terms exist, generally speaking, one can only speak about stability in a bounded domain of initial perturbations. If, however, the matrices A_1 and N_1 are obviously independent of x_1 and the matrix N_1 is also independent of x_2, the right-hand sides of equality (2.3) do not contain higher derivatives. In this case, the vectors Q_1 and Q_2 ensure the conditions for the asymptotic stability of the unperturbed state of the system "on the whole".

In spite of the fact that the vectors x_1 and x_2 are contained in A_1 and N_1, the following scheme can be proposed to construct Q_1 and Q_2 such that the unperturbed state of the system is stable "on the whole".

We multiply Eqs (1.1) by the non-singular matrices AA_1^{-1} and NN_1^{-1} respectively, where $A(x_1, x_2, t)$ and $N(x_1, t)$ are arbitrary bounded positive-definite symmetric matrices, with bounded derivatives with respect to all variables for all $x_1, x_2, t \geqslant t_0$, which satisfy generalized Silvester's criteria. Then, when the "generalized forces" of the controls Q_1 and Q_2 are introduced into the right-hand side, Eqs (1.1) take the form

$$A(x_1,x_2,t)\ddot{x}_1=AA_1^{-1}B_1+AA_1^{-1}Q_1 \\ N(x_1,t)\dot{x}_2=NN_1^{-1}K_1+NN_1^{-1}Q_2$$

Multiplying the first equation scalarly by y and the second equation by x_2, we obtain an analogue of Eq. (2.2) from which it follows that, in the case of the values

$$Q_1 = -B_1 + A_1 A^{-1}\left[-Dy - F_1 x_1 + A\frac{\partial f}{\partial x_1}y + A\left(\frac{\partial f}{\partial x_1}f + \frac{\partial f}{\partial t}\right) - \frac{1}{2}\frac{dA(x_1,x_2,t)}{dt}y\right]$$
$$Q_2 = -K_1 + N_1 N^{-1}\left[-F_2 x_2 - \frac{1}{2}\frac{dN(x_1,t)}{dt}x_2\right] \tag{2.5}$$

we obtain an equation of the form of (2.4).

When the controls are constructed in this way, the conditions for the existence and uniqueness of the solutions of system (1.1) are not affected in the whole of the phase space for all $t \geqslant t_0$.

3. IMPROVEMENT OF THE QUALITY OF THE TRANSIENT OF A MECHANICAL SYSTEM BY OPTIMAL CONTROL

Consider the mechanical system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q}\right) - \frac{\partial T}{\partial q} = Q(q,\dot q,t), \quad T = \frac{1}{2}\dot q^T A(q,t)\dot q + \dot q^T a(q,t) + \frac{1}{2}a_0(q,t)$$

It has been shown in [2] that, for the choice $f = H(t)x$ and the generalized force vector Q in the form

$$Q = -Dy - Fx + \frac{1}{2}A'y - \frac{\partial T_2}{\partial x} + \frac{\partial F_2'}{\partial x} + \frac{1}{2}\frac{\partial A(q_0+x,t)}{\partial t}y - GHx + \frac{\partial b}{\partial t} - \frac{1}{2}\frac{\partial b_0}{\partial x} + \frac{dAHx}{dt} \tag{3.1}$$

the system possesses a preset motion $q_0(t)$ which is stabilized "on the whole".

Here

$$y = \dot x - H(t)x, \quad T_2 = \frac{1}{2}\dot q^T A\dot q T_2' = \frac{1}{2}y^T Ay$$
$$b(x,t) = A(q_0+x,t)\dot q_0 + a(q_0+x,t), \quad b_0 = \dot q_0^T A\dot q_0 + 2\dot q_0^T a + a_0$$

A' is an $n \times n$-matrix with the elements $a_{ij}' = x^T H^T \partial a_{ij}/\partial x$, a_{ij} is an element of the matrix A, $H(t)$ is an arbitrary bounded $n \times n$-matrix with a bounded and continuous derivative with respect to t, G is a skew symmetric matrix with elements $g_{\nu i} = -\partial b_\nu/\partial x_i + \partial b_i/\partial x_\nu$, b_ν, and b_i are elements of the vector b, and D and F are positive-definite symmetric matrices.

Here, for the deviations $x = q - q_0(t)$ from the preset motion, we have the Lyapunov function and its time derivative

$$V = T_2' + x^T Fx/2, \quad \dot V = -y^T Dy + x^T(H^T F + \dot F/2)x \tag{3.2}$$

where $(H^T F + F/2)$ is a negative-definite matrix.

In the case of the non-linear substitution $y = \dot x - f(x,f)$, using a non-linear function $f(x, t)$ with bounded and differentiable elements, which admits of an infinitely small higher limit and with the choice of the generalized force in a form which differs from (3.1) in that the Hx is replaced by $f(x, t)$, we obtain

$$V = T_2' + x^T Fx/2, \quad \dot V = -y^T Dy + f^T(x,t)Fx + x^T \dot Fx/2$$

If the function $f(x, t)$ and the matrix F are chosen such that the function $f^T Fx + x^T Fx/2$ is negative-definite, the quality of the transient will have the following value

$$\int_{t_0}^{\infty}\left[y^T Dy - \left(f^T Fx + \frac{1}{2}x^T \dot Fx\right)\right]dt = V_0, \quad V_0 = V(t_0) \tag{3.3}$$

In order to improve the quality of the transient, we add an additional component $M(x, \dot x, t)u$ to the vector Q, where M is an $n \times r$-matrix and u is an r-dimensional control vector. Following the approach proposed earlier [4], we seek the integrand of the functional for the quality of the optimal control in the form

$$W^0 = F_1(x, \dot{x}, t) + u^T R(x, \dot{x}, t)u$$

where R is a positive-definite bounded symmetric $r \times r$-matrix.

We now construct the function [5]

$$B^0 = \dot{V} + u^T M^T y + F_1 + u^T Ru$$

From the condition $\partial B^0/\partial u = 0$, we obtain $u^0 = -R^{-1}M^T y/2$ and, from the condition $B^0 = 0$, we have

$$F_1 = -\dot{V} + y^T MR^{-1}M^T y/4$$

At the same time, we have

$$W^0 = -\dot{V} + y^T MR^{-1}M^T y/2$$

and the following criterion for the quality of the transient

$$\int_{t_0}^{\infty} \left[y^T Dy - \left(f^T Fx + \frac{1}{2} x^T \dot{F} x \right) + \frac{1}{2} y^T MR^{-1}M^T y \right] dt = V_0$$

where V_0 is the value of $V = T_2' + x^T Fx/2$ when $t = t_0$.

The addition of the optimal control to Q has therefore improved the quality of the transient since the magnitude of the quality functional when $u = 0$ (the left-hand side of (3.3)) has been reduced by

$$\frac{1}{2} \int_{t_0}^{\infty} y^T MR^{-1}M^T y dt$$

If there is no need to require that the functional

$$\int_{t_0}^{\infty} W^0 dt$$

should be a minimum, the function F_1 can then be chosen from the condition

$$0 \leqslant F_1 \leqslant -\dot{V} + y^T MR^{-1}M^T y/4$$

In this case, a guaranteed estimated of the quality of the transient

$$\int_{t_0}^{\infty} W^0 dt \leqslant V_0$$

is attained.

Note that, if the problem is solved within the framework of the equations of the first approximation, the terms

$$\left(\frac{1}{2} A' y - \frac{\partial T_2}{\partial x} + \frac{\partial T_2'}{\partial x} \right)$$

of the second order of smallness in the expression for Q can be equated to zero. In this case, if of the matrices $H = (\partial f/\partial x)_0 D, F$ it is required that $H = H^T$ and the matrices

$$\left(D - \frac{1}{2} \frac{\partial A(q_0, t)}{\partial t} + AH \right), \left(F - AH^2 - \frac{dAH}{dt} \right)$$

should be positive-definite then, subject to the condition that the matrix

$$-(H^T F' + \dot{F}'/2), \quad \text{where} \quad F' = F - dAH/dt - AH^2$$

should be positive-definite the vector Q can be constructed in the form

$$Q = -Dy - Fx + \frac{\partial b}{\partial x} - \frac{1}{2}\frac{\partial b_0}{\partial x} - GHx$$

4. EXAMPLES

1. *Stabilization of the orientation of a satellite in an elliptic orbit.* When the gravitational moment is used to stabilize the rotational motion of the orientation of a satellite, the following equation of motion holds [7]

$$\ddot{\alpha} - l(\nu)\dot{\alpha} + m(\nu)\sin\alpha = 2l(\nu) + u$$

where

$$l(\nu) = \frac{2\varepsilon\sin\nu}{1+\varepsilon\cos\nu}, \quad m(\nu) = \frac{n^2}{1+\varepsilon\cos\nu}, \quad n^2 = 3\frac{A-C}{B}, \quad \alpha = 2\Theta$$

Θ is the angle of deviation of the OZ axis of the satellite from the radial position, ν is the true anomaly, ε is the orbit eccentricity ($0 < \varepsilon < 1$) and derivatives with respect to ν are denoted by dots. We take the control vector u in the form of the sum

$$u = u_0(\nu, \alpha, \dot{\alpha}) + M$$

where u_0 is the control moment in the case of the unperturbed motion and M is the stabilizing moment.

In this case, the equation of the perturbed motion has the form [6]

$$\ddot{x} = -l(\nu)\dot{x} - g(\nu, x)\sin\frac{x}{2} + M, \quad g(\nu, x) = 2m(\nu)\cos(\alpha_0 + \frac{x}{2})$$

We make the substitution

$$y = \dot{x} + Hx, \qquad H = \text{const} > 0$$

On replacing $\ddot{x}$ by $\ddot{x} = \dot{y} - Hy + H^2x$, we obtain

$$\dot{y} = -l(\nu)(y - Hx) - H^2x + Hy - g(\nu, x)\sin\frac{x}{2} + M$$

We multiply this equation by y and choose M in the form

$$M = -D_1y - F_1x$$

and introduce the notation

$$D = D_1 - H + l(\nu), \quad F = F_1 + H^2 - Hl(\nu) + \frac{g(\nu, x)}{x}\sin\frac{x}{2}$$

After replacing y in the expression (yFx) and transferring (xFx) to the left-hand side, we obtain

$$\frac{1}{2}\frac{d}{d\nu}(y^2 + Fx^2) = -Dy^2 - \left(FH - \frac{\dot{F}}{2}\right)x^2$$

If it is required that the conditions

$$D > 0, \quad F > 0, \quad FH - \frac{\dot{F}}{2} > 0$$

be satisfied, then, in the case of control moment (4.1), the unperturbed motion will be stabilized on the whole. Here, the quality of the transient is determined by the equality

$$\int_{t_0}^{\infty}\left[Dy^2+\left(FH-\frac{\dot F}{2}\right)\right]dt = y_0^2+Fx_0^2;\ y_0=y(t_0), x_0=x(t_0) \tag{4.2}$$

2. *Stabilization of the programmed motion of a mathematical pendulum with a moving point of suspension.* Suppose the point of suspension of a pendulum executes arbitrarily specified planar motions, which are described by the laws $\Theta(t)$ and $\nu(t)$, along the vertical and horizontal axes of a fixed system of coordinates.

When

$$m=1,\quad l=1/a,\quad a>0$$

where l is the length of the pendulum and m is the mass, the following equation of motion [8] holds

$$\ddot\varphi + a\ddot\nu(t)\cos\varphi + a\ddot\Theta\sin\varphi + ag\sin\varphi = M_1$$

We substitute $\varphi = \varphi_0(t) + x$, where φ_0 is the specified programme and we represent the control moment in the form of the sum

$$M_1 = M_0 + M,\quad M_0 = \ddot\varphi_0 + a\ddot\nu\cos(x+\varphi_0, t)$$

where M is the stabilizing moment. We then obtain the equation of the perturbed motion in the form

$$\ddot x + k\sin(x+\varphi_0) = M,\quad k = a(\ddot\Theta + g)$$

This equation can be represented in the form

$$\ddot x = -g(x,t)\sin\frac{x}{2} + M,\quad g = 2k\cos\left(\varphi_0 + \frac{x}{2}\right)$$

Hence, we obtain an equation which is a special case of the equation of the perturbed motion obtained in Example 1 when $l=0$. Consequently, the equality

$$\frac{1}{2}\frac{d}{dt}(y^2+Fx^2) = -Dy^2 - \left(FH - \frac{\dot F}{2}\right)x^2$$

$$D = D_1 - H,\quad F = F_1 + H^2 + \frac{g(x,t)}{x}\sin\frac{x}{2},\quad y = \dot x + Hx,\quad H = \text{const} > 0$$

holds, where D_1 and F_1 are elements of the stabilizing moment (4.1) while satisfy the conditions $D > 0, F > 0$.

The quality of the transient (4.2) is attained in the case of stabilization with such a moment.

3. *Stabilization of the programmed motion of a gyropendulum on a moving base.* Suppose a gyropendulum, which has an angular spin velocity equal to $\zeta_0 = \zeta(t)$, is placed on a platform which executes vertical motion in accordance with the law $\zeta\eta\zeta$. The position of the OZ axis of symmetry of the gyropendulum relative to a fixed system of coordinates $\zeta\eta\zeta$ (the $O\zeta$ axis is the vertical axis) is specified by the angles α and β [3, p. 56]. Here, the kinetic energy T and the potential energy Π of the gyropendulum take the form

$$T = \frac{1}{2}A(\dot\alpha^2 + \dot\beta^2\cos^2\alpha) + \frac{1}{2}C(\dot\varphi + \dot\beta\sin\alpha)^2 -$$

$$-Ml\dot\zeta_0(\dot\alpha\sin\alpha\cos\beta + \dot\beta\cos\alpha\sin\beta) + \frac{1}{2}M\dot\zeta_0^2$$

$$\Pi = Mg\zeta + Mgl\cos\alpha\cos\beta$$

where M is the mass, l is the distance from the point of support to the centre of gravity, A is the moment of inertia about the OX and OY axis and C is the moment of inertia about the OZ axis.

We shall describe the motion of the gyropendulum by means of the Lagrange equations

$$\frac{d}{dt}\frac{\partial T_2}{\partial \dot{q}} - \frac{\partial T_2}{\partial q} = -\frac{\partial \Pi}{\partial q} + Q; \ q_1 = \alpha, \ q_2 = \beta, \ q_3 = \varphi$$

In order to obtain the equations of the perturbed motion, we replace q by $q_0(t) + x$ and obtain

$$T = T_2 + T_1 + T_0$$

$$T_2 = \frac{1}{2}A(\dot{x}_1^2 + \dot{x}_2^2\cos^2\alpha) + \frac{1}{2}C(\dot{x}_3^2 + \dot{x}_2^2\sin^2\alpha) - C\dot{x}_2\dot{x}_3\sin\alpha$$

$$T_1 = \dot{x}^T b(x,t), \ T_0 = \frac{1}{2}b_0(x,t)$$

$$b = (b_1, b_2, b_3)$$

$$b_1 = A\dot{\alpha}_0 - Ml\dot{\zeta}_0\sin(\alpha_0 + x_1)\cos(\beta_0 + x_2)$$

$$b_2 = [A\cos^2(\alpha_0 + x_1) + C\sin(\alpha_0 + x_1)]\dot{\beta}_0 - C\dot{\varphi}_0\sin(\alpha_0 + x_1) -$$
$$-Ml\dot{\zeta}_0\cos(\alpha_0 + x_1)\sin(\beta_0 + x_2)$$

$$b_3 = -C\dot{\beta}_0\sin(\alpha_0 + x_1) + C\dot{\varphi}_0$$

$$b_0 = A[\dot{\alpha}_0^2 + \dot{\beta}_0^2\cos^2(\alpha_0 + x_1)] + C[\dot{\varphi}_0 - \dot{\beta}_0\sin(\alpha_0 + x_1)]^2 -$$
$$-2Ml\dot{\zeta}_0[\dot{\alpha}_0\sin(\alpha_0 + x_1)\cos(\beta_0 + x_2) +$$
$$+\dot{\beta}_0\cos(\alpha_0 + x_1)\sin(\beta_0 + x_2)] + M\dot{\zeta}_0^2$$

We note that

$$a_{11} = A, \ a_{12} = a_{13} = a_{21} = a_{31} = 0, \ a_{22} = A\cos^2(\alpha_0 + x_1) + C\sin^2(\alpha_0 + x_1)$$
$$a_{23} = a_{32} = -C\sin(\alpha_0 + x_1), a_{33} = C$$

are the elements of the 3×3-matrix A^0 of the quadratic form T_2.

We choose the three-dimensional vector of the generalized forces in the form (3.1) with the addition of the term $\partial\Pi/\partial q$ to the right-hand side and with of A replaced by A^0. In this case, the programmed motion

$$\alpha = \alpha_0(t), \quad \beta = \beta_0(t), \quad \varphi = \varphi_0(t)$$

occurs and it is stabilized as a whole. The quantities occurring in Q, which have not been indicated above, have the following expressions

$$T_2' = \frac{1}{2}A(y_1^2 + y_2^2\cos^2\alpha) + \frac{1}{2}C(y_3^2 + y_2^2\sin^2\alpha) - Cy_2y_3\sin\alpha, \ \alpha = \alpha_0 + x_1$$

A' is a 3×3-matrix with elements a_{ij}', which are defined in terms of the elements a_{ij} of the matrix A^0 in the form $a_{ij}' = x^T H^T \partial a_{ij}/\partial x, y = x - H(t)x, H(t)$ is an arbitrary 3×3-matrix with bounded and continuous elements, G is a skew symmetric 3×3-matrix with elements $g_{ij} = -\partial b_\nu/\partial x_i + \partial b_i/\partial x_\nu$, b_ν and b_i are elements of the vector b, and D and F are positive-definite symmetric matrices.

With this choice of Q, we have the Lyapunov function and its time derivative in the form (3.2), where $H(t)$ is chosen such that the matrix $(H^T F + F/2)$ is negative-definite. In this case, we obtain the following quality of the transient

$$\int_{t_0}^{\infty}\left[y^T Dy - x^T\left(H^T F + \frac{\dot{F}}{2}\right)x\right]dt = V_0, \ V_0 = V(t_0) \tag{4.3}$$

This quality can be improved by adding the additional component u^0 to Q and requiring that the functional of it

$$\int_{t_0}^{\infty}[-V+(u^0)^2]dt$$

should attain a minimum value.

As in Section 3, we obtain that $u^0 = -y/2$ for which, in the case of the improved quality, we have a relation which differs from (4.3) in the addition of the term $y^2/2$ to the integrand. Note that the value of the quality functional when $u^0 = 0$ is reduced by

$$\frac{1}{2}\int_{t_0}^{\infty} y^2 dt$$

We will now consider one important special case when the vertical position of the axis of the gyropendulum $\alpha_0 = \dot{\alpha}_0 = \beta_0 = \dot{\beta}_0 = 0$ is the unperturbed state in the case of a constant spin velocity around the OZ axis ($\varphi = \omega = \text{const}$). Then, the position of the OZ axis is defined by the generalized coordinates $q_1 = \alpha$, $q_2 = \beta$.

In this case

$$T_2 = \frac{1}{2}A(\dot{x}_1^2 + \dot{x}_2^2\cos^2 x_1) + \frac{1}{2}C\dot{x}_2^2\sin^2 x_1$$

$$T_1 = \dot{x}_1 b_1 + \dot{x}_2 b_2,\ T_0 = \frac{1}{2}(C\omega^2 + M\dot{\zeta}_0^2),\ b_0 = C\omega^2 + M\dot{\zeta}_0^2,$$

$$T_2' = \frac{1}{2}A(y_1^2 + y_2^2\cos^2 x_1) + \frac{1}{2}Cy_2^2\sin^2 x_1$$

$$b_1 = -Ml\dot{\zeta}_0 \sin x_1 \cos x_2,\ b_2 = -C\omega \sin x_1 - Ml\dot{\zeta}_0 \cos x_1 \sin x_2$$

$$g_{12} = -g_{21} = C\omega \cos x_1$$

The elements of the matrix of the quadratic from T_2 have the form

$$a_{11} = A,\ a_{22} = A\cos^2 x_1 + C\sin^2 x_1,\ a_{12} = a_{21} = 0$$

Consequently, the equalities

$$\frac{\partial A^0}{\partial t} = 0,\ \frac{\partial b_0}{\partial x} = 0,\ \frac{\partial b}{\partial t} = \begin{Vmatrix} -Ml\ddot{\zeta}_0 \sin x_1 \cos x_2 \\ -Ml\ddot{\zeta}_0 \cos x_1 \sin x_2 \end{Vmatrix}$$

hold.

We select $H = -I$, where I is the identity matrix. Then

$$y = \dot{x} + Ix$$

$$\frac{dA^0 Hx}{dt} = -A^0\dot{x} - \dot{A}^0 x = \begin{Vmatrix} -A\dot{x}_1 \\ -(A\cos^2 x_1 + C\sin^2 x_1)\dot{x}_2 - (C-A)\dot{x}_1 x_2 \end{Vmatrix}$$

$$\frac{\partial a_{11}}{\partial x} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix},\ \frac{\partial a_{22}}{\partial x} = \begin{Vmatrix} (C-A)\sin 2x_1 \\ 0 \end{Vmatrix},\ a_{11}' = 0$$

$$a_{22}' = (A-C)x_1 \sin 2x_1,\ A'y = \begin{Vmatrix} 0 \\ (A-C)x_1 y_2 \sin 2x_1 \end{Vmatrix}$$

$$\frac{\partial T_2}{\partial x} = \begin{Vmatrix} \frac{1}{2}(C-A)\dot{x}_2^2 \sin 2x_1 \\ 0 \end{Vmatrix},\ \frac{\partial T_2'}{\partial x} = \begin{Vmatrix} \frac{1}{2}(C-A)y_2^2 \sin 2x_1 \\ 0 \end{Vmatrix}$$

$$\frac{\partial \Pi}{\partial q} = \begin{Vmatrix} -Mgl\sin x_1 \cos x_2 \\ -Mgl\cos x_1 \sin x_2 \end{Vmatrix}, \quad G = \begin{Vmatrix} 0 & -C\omega\cos x_1 \\ C\omega\cos x_1 & 0 \end{Vmatrix}$$

If the problem is solved in the first approximation, we have

$$Q = \begin{Vmatrix} -Mglx_1 \\ -Mglx_2 \end{Vmatrix} - D\begin{Vmatrix} \dot{x}_1 + x_1 \\ \dot{x}_2 + x_2 \end{Vmatrix} - F\begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} -$$

$$-\begin{Vmatrix} Ml\ddot{\zeta}_0 x_1 \\ Ml\ddot{\zeta}_0 x_2 \end{Vmatrix} + \begin{Vmatrix} A\dot{x}_1 \\ A\dot{x}_2 \end{Vmatrix} + \begin{Vmatrix} -C\omega x_2 \\ +C\omega x_1 \end{Vmatrix}$$

If we choose D and F in the form of diagonal constant matrices with positive elements d_1, d_2 and f_1, f_2 respectively, we obtain

$$Q_1 = -x_1(Mgl + d_1 + f_1 + Ml\ddot{\zeta}_0) - C\omega x_2 - \dot{x}_1(d_1 - A)$$

$$Q_2 = C\omega x_1 - x_2(Mgl + d_2 + f_2 + Ml\ddot{\zeta}_0) - \dot{x}_2(d_2 - A)$$

In this case, the Lyapunov function and its time derivative are expressed in the form

$$V = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(f_1x_1^2 + f_2x_2^2)$$

$$\dot{V} = -d_1y_1^2 - d_2y_2^2 - f_1x_1^2 - f_2x_2^2$$

Consequently, the following quality for the transient holds

$$\int_{t_0}^{\infty}(d_1y_1^2 + d_2y_2^2 + f_1x_1^2 + f_2x_2^2)dt = V_0, \quad V_0 = V(t_0) \tag{4.4}$$

If the component u^0 of the optimal control is added to the vector Q in order to improve the quality of the transient then, following what has been stated in Section 3, we have $u^0 = -y/2$ and, also, an improved quality for the transient which differs from (4.4) in the addition of the term $(y_1^2 + y_2^2)/2$ to the integrand.

In this case, the magnitude of the quality when $u^0 = 0$ is improved by an amount

$$\frac{1}{2}\int_{t_0}^{\infty}(y_1^2 + y_2^2)dt$$

I wish to thank V. V. Rumyantsev for his interest.
This research was supported financially by the Russian Foundation for Basic Research (99-01-01193).

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Translated by E.L.S.